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New Fixed Point Theorems for Mappings in Metric and Fuzzy Metric Spaces: An Analytical Approach

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ABSTRACT

The Banach Contraction Principle remains one of the most powerful tools of nonlinear functional analysis, and its extensions to more general contractive conditions and to fuzzy metric settings continue to attract active research interest. This study develops a unified analytical framework for fixed point theory in complete metric spaces and complete fuzzy metric spaces (in the sense of George and Veeramani). A new generalized rational-type contractive condition is introduced, combining a linear term, a rational term, and a mixed term, and existence and uniqueness of a fixed point is established together with convergence of the associated Picard iteration. An analogous condition is then formulated in the fuzzy metric setting through a monotone transformation of the membership function, and a parallel existence-uniqueness theorem is proved. A common fixed point theorem for a weakly compatible pair of self-mappings is further obtained as an extension. The theoretical results are illustrated through worked numerical examples, iteration tables, and graphical depictions of convergence, and are compared against classical conditions of Banach, Kannan, and Chatterjea in a summary table. Potential applications to integral equations and to fuzzy-uncertainty models are discussed, and directions for further generalization are indicated.

Keywords: Fixed point theorem; complete metric space; fuzzy metric space; rational contraction; Picard iteration; continuous t-norm; common fixed point; George–Veeramani space.

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Introduction

Fixed point theory occupies a central position in nonlinear functional analysis because of its capacity to convert questions about the existence and uniqueness of solutions of equations into questions about the existence and uniqueness of points left invariant by a mapping. The cornerstone of the subject is the Banach Contraction Principle (Banach, 1922), which guarantees that every contraction mapping on a complete metric space has a unique fixed point, obtainable as the limit of the Picard iteration generated from any starting point. Because of its constructive nature, the principle has found application across differential equations, integral equations, optimization theory, economics, and numerical analysis.

Since Banach's original theorem, a large body of literature has been devoted to weakening or reformulating the contractive condition. Kannan (1968) proved a fixed point theorem for mappings that need not be continuous, using a condition expressed purely in terms of the displacement of each point under the mapping. Chatterjea (1972) proposed a symmetric analogue of Kannan's condition. Subsequent authors, including Reich, Hardy and Rogers, and Ćirić, introduced rational-type and combined contractive conditions that unify and generalize the earlier results by incorporating several distance terms simultaneously, often permitting a wider class of mappings to be covered by a single theorem.

In a parallel development, Zadeh (1965) introduced the notion of a fuzzy set to model gradations of membership rather than the classical binary notion of belonging. This idea was subsequently used by Kramosil and Michálek (1975) to define fuzzy metric spaces, generalizing the classical concept of distance to a membership function that measures the degree of nearness between two points relative to a parameter t. George and Veeramani (1994) modified these axioms to obtain a Hausdorff topology on the induced space, and this refined formulation is now the standard framework used in the fixed point literature. Grabiec (1988) was the first to establish a fuzzy analogue of the Banach Contraction Principle within this setting, and this line of work has since been extended by several authors, including Gregori and Sapena (2002), who studied fuzzy contractive mappings of Banach type more systematically.

The present study addresses this gap with the following objectives:

- To introduce a new generalized rational-type contractive condition on a complete metric space that combines a linear term, a rational term, and a mixed (Hardy–Rogers type) term, and to establish an existence and uniqueness theorem (Theorem 4.1) together with convergence of the Picard iteration.
- To formulate an analogous condition in a complete fuzzy metric space (George–Veeramani sense) using a monotone transformation of the membership function, and to prove a corresponding existence and uniqueness theorem (Theorem 4.2).
- To extend the fuzzy result to a common fixed point theorem for a weakly compatible pair of self-mappings (Theorem 4.3).
- To substantiate the theoretical results with worked numerical examples, iteration tables, and convergence graphs, and to situate the new condition relative to classical conditions through a comparative table.
- To indicate representative applications and directions for further generalization.

2. Preliminaries and Basic Definitions

This section collects the definitions and classical results that are used in the sequel. Standard references are Banach (1922), Kannan (1968), Chatterjea (1972), Zadeh (1965), Kramosil and Michálek (1975), George and Veeramani (1994), and Grabiec (1988).

2.1 Metric Spaces

Definition 2.1 (Metric space). Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a metric on X if for all x, y, z ∈ X: (i) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$; (ii) $d(x,y) = d(y,x)$; (iii) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality). The pair (X, d) is then called a metric space.

Definition 2.2 (Cauchy sequence and completeness). A sequence $\{x_n\}$ in (X, d) is called a Cauchy sequence if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. The space (X, d) is complete if every Cauchy sequence in X converges to a point of X.

Definition 2.3 (Contraction mapping). A self-map $T : X \to X$ is called a contraction (Banach contraction) if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq k \cdot d(x,y)$ for all x, y ∈ X.

$$d(Tx, Ty) \leq k \cdot d(x,y), \quad 0 \leq k < 1 \qquad (2.1)$$

Theorem 2.1 (Banach, 1922). If (X, d) is a complete metric space and $T : X \to X$ is a contraction, then T has a unique fixed point $x^* \in X$, and for any $x_0 \in X$ the Picard sequence $x_{n+1} = Tx_n$ converges to x^*. This classical theorem is the starting point for all the generalizations developed below.

2.2 Kannan and Chatterjea Contractions

Definition 2.4 (Kannan contraction, 1968). $T : X \to X$ is a Kannan mapping if there exists $\beta \in [0, 1/2)$ such that $d(Tx,Ty) \leq \beta[d(x,Tx) + d(y,Ty)]$ for all x, y ∈ X.

Definition 2.5 (Chatterjea contraction, 1972). $T : X \to X$ is a Chatterjea mapping if there exists $\gamma \in [0, 1/2)$ such that $d(Tx,Ty) \leq \gamma[d(x,Ty) + d(y,Tx)]$ for all x, y ∈ X.

These two conditions are, in general, independent of the Banach condition: neither implies the other, and mappings satisfying one need not be continuous, unlike Banach contractions. This independence motivates the search for combined conditions that cover all three as special cases, which is undertaken in Section 4.

2.3 Fuzzy Sets and Fuzzy Metric Spaces

Definition 2.6 (Fuzzy set, Zadeh 1965). A fuzzy set A on a non-empty set X is a function $A : X \to [0,1]$. The value A(x) is interpreted as the degree of membership of x in A.

Definition 2.7 (Continuous t-norm). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it is commutative, associative, continuous, satisfies $a * 1 = a$ for all $a \in [0,1]$, and is non-decreasing in each argument. Common examples are the minimum t-norm $a * b = \min(a,b)$ and the product t-norm $a * b = a \cdot b$.

Definition 2.8 (Fuzzy metric space, George & Veeramani 1994). A triple (X, M, *) is called a fuzzy metric space if X is a non-empty set, * is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying, for all $x, y, z \in X$ and $s, t > 0$: (FM1) $M(x,y,t) > 0$; (FM2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$; (FM3) $M(x,y,t) = M(y,x,t)$; (FM4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$; (FM5) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

$$M(x,y,t) * M(y,z,s) \le M(x,z,t+s) \quad \text{(2.2 (FM4))}$$

A canonical example, used repeatedly in Section 5, is the standard fuzzy metric induced by an ordinary metric d:

$$M(x,y,t) = t / (t + d(x,y)), \quad t > 0 \quad (2.3)$$

with the product t-norm $a * b = a \cdot b$. It is routine to check that (FM1)–(FM5) hold for this M, so that (X, M, *) is a fuzzy metric space whenever (X, d) is a metric space.

Definition 2.9 (Convergence, Cauchy sequence, completeness in FM-spaces). A sequence $\{x_n\}$ in (X, M, *) converges to x if $M(x_n, x, t) \to 1$ as $n \to \infty$, for every $t > 0$. It is a Cauchy sequence if, for every $\varepsilon \in (0,1)$ and $t > 0$, there exists N such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \ge N$. (X, M, *) is complete if every Cauchy sequence converges in X.

Theorem 2.2 (Grabiec, 1988). Let (X, M, *) be a complete fuzzy metric space in the Kramosil–Michálek sense with $\lim_{t\to\infty} M(x,y,t) = 1$, and suppose $T : X \to X$ satisfies $M(Tx, Ty, kt) \ge M(x,y,t)$ for some $k \in (0,1)$ and all $x, y \in X$, $t > 0$. Then T has a unique fixed point. This result is the fuzzy analogue of the Banach principle and is the direct precursor of Theorem 4.2 proved below.

3. Related Work: Classical Contraction Principles

Table 1 summarizes the chronological development of the classical contraction conditions that motivate the generalized condition proposed in Section 4. Each subsequent condition either weakens the hypothesis on T (Kannan, Chatterjea) or broadens the class of admissible inequalities by combining several distance terms (Reich, Hardy–Rogers, Ćirić), and the fuzzy line of development (Kramosil–Michálek, George–Veeramani, Grabiec) transports the same idea to a setting in which nearness is graded rather than binary.

Author(s) / Year	Space	Core Contractive Idea	Remarks
Banach (1922)	Metric	$d(Tx,Ty) \le k\, d(x,y)$, $k<1$	Guarantees uniqueness and Picard convergence
Kannan (1968)	Metric	$d(Tx,Ty) \le \beta[d(x,Tx)+d(y,Ty)]$	T need not be continuous
Chatterjea (1972)	Metric	$d(Tx,Ty) \le \gamma[d(x,Ty)+d(y,Tx)]$	Symmetric analogue of Kannan
Zadeh (1965)	Fuzzy sets	Membership function $A: X\to[0,1]$	Foundation of fuzzy mathematics
Kramosil & Michálek (1975)	Fuzzy metric	Statistical-metric-style axioms	First fuzzy metric axioms
George & Veeramani (1994)	Fuzzy metric	Refined FM1–FM5 axioms	Induces Hausdorff topology
Grabiec (1988)	Fuzzy metric	$M(Tx,Ty,kt) \ge M(x,y,t)$	Fuzzy analogue of Banach

Table 1. Chronological summary of classical fixed point conditions.

A recurring observation in this literature is that the rational and combined conditions (of Reich and Ćirić type) are strictly more general than the Banach condition alone, in the sense that they admit mappings that are not Banach contractions but still possess a unique fixed point. This observation is the direct motivation for Definition 4.1 below, which combines a linear term, a rational term evaluated at the images of x and y, and a symmetric mixed term into a single inequality.

4. Main Results

This section introduces the new generalized contractive condition and its fuzzy analogue, and establishes the existence–uniqueness theorems that are the central contribution of the study.

4.1 A Generalized Rational Contractive Condition

Definition 4.1. Let (X, d) be a metric space. A self-map $T : X \to X$ is said to satisfy the generalized rational contractive condition if there exist non-negative constants α, β, γ with $\alpha + \beta + 2\gamma < 1$ such that, for all $x, y \in X$,

$$d(Tx,Ty) \le \alpha \cdot d(x,y) + \beta \cdot [d(x,Tx) \cdot d(y,Ty)]/[1+d(x,y)] + \gamma \cdot [d(x,Ty)+d(y,Tx)] \quad (4.1)$$

Condition (4.1) reduces to the Banach condition (2.1) when $\beta = \gamma = 0$ (with $k = \alpha$), and it reduces to a symmetric Chatterjea-type condition when $\alpha = \beta = 0$ (with γ replacing the constant of Definition 2.5, subject to $\gamma < 1/2$, which is implied by $\alpha + \beta + 2\gamma < 1$). The rational middle term is bounded by $\min\{d(x,Tx), d(y,Ty)\}$ for all $x \neq y$, so it vanishes automatically whenever x or y is a fixed point of T, which is essential in the uniqueness argument below.

Theorem 4.1. Let (X, d) be a complete metric space and let $T : X \to X$ satisfy the generalized rational contractive condition (4.1). Then:

- T has a fixed point $x^* \in X$;
- x^* is the unique fixed point of T in X; and
- for every $x_0 \in X$, the Picard sequence $x_{n+1} = Tx_n$ converges to x^*, with the a priori error estimate $d(x_n, x^*) \leq [h^n/(1-h)]\cdot d(x_0,x_1)$, where $h = (\alpha+\gamma)/(1-\beta-\gamma)$.

Proof. Fix $x_0 \in X$ and define the Picard sequence $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$. Write $d_n = d(x_n, x_{n+1})$. Applying (4.1) with $x = x_{n-1}$, $y = x_n$ gives

$$d_n = d(Tx_{n-1},Tx_n) \leq \alpha\cdot d(x_{n-1},x_n) + \beta\cdot[d(x_{n-1},x_n)\cdot d(x_n,x_{n+1})]/[1+d(x_{n-1},x_n)] + \gamma\cdot[d(x_{n-1},x_{n+1})+d(x_n,x_n)] \quad (4.2)$$

Since the fraction $d(x_{n-1},x_n)/[1+d(x_{n-1},x_n)]$ never exceeds 1, the middle term on the right of (4.2) is bounded above by $\beta\cdot d_n$. Since $d(x_n,x_n) = 0$ and, by the triangle inequality, $d(x_{n-1},x_{n+1}) \leq d_{n-1} + d_n$, inequality (4.2) yields

$$d_n \leq \alpha\cdot d_{n-1} + \beta\cdot d_n + \gamma\cdot(d_{n-1}+d_n) \quad (4.3)$$

Collecting the d_n terms on the left gives $d_n\cdot(1 - \beta - \gamma) \leq (\alpha+\gamma)\cdot d_{n-1}$. Because $\alpha+\beta+2\gamma < 1$, we have $\beta+\gamma < 1$, so the coefficient $(1-\beta-\gamma)$ is strictly positive and division is legitimate. Setting $h = (\alpha+\gamma)/(1-\beta-\gamma)$, the hypothesis $\alpha+\beta+2\gamma<1$ gives $(\alpha+\gamma) < 1-\beta-\gamma$, so $h \in [0,1)$. We obtain the geometric recursion

$$d_n \leq h\cdot d_{n-1} \leq h^2\cdot d_{n-2} \leq \cdots \leq h^n\cdot d_0, \quad d_0 = d(x_0,x_1) \quad (4.4)$$

For $m > n$, repeated use of the triangle inequality together with (4.4) gives

$$d(x_n,x_m) \leq d_n + d_{n+1} + \cdots + d_{m-1} \leq (h^n+h^{n+1}+\cdots+h^{m-1})\cdot d_0 \leq [h^n/(1-h)]\cdot d_0 \quad (4.5)$$

Since $0 \leq h < 1$, the right side of (4.5) tends to 0 as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, $x_n \to x^*$ for some $x^* \in X$. Passing to the limit as $n \to \infty$ in (4.5) with m fixed and then letting $m \to \infty$ gives the error estimate stated in part (c).

To show that x^* is a fixed point, apply (4.1) with $x = x_n$, $y = x^*$ (for $x_n \neq x^*$, discarding at most finitely many terms where $x_n = x^*$, in which case $Tx^* = x_{n+1} \to x^*$ trivially):

$$d(Tx_n,Tx^*) \leq \alpha\cdot d(x_n,x^*) + \beta\cdot[d(x_n,Tx_n)\cdot d(x^*,Tx^*)]/[1+d(x_n,x^*)] + \gamma\cdot[d(x_n,Tx^*)+d(x^*,Tx_n)] \quad (4.6)$$

As $n \to \infty$, $x_n \to x^*$, $Tx_n = x_{n+1} \to x^*$, and $d(x_n,Tx_n) = d_n \to 0$, so the right side of (4.6) tends to $\gamma\cdot d(x^*,Tx^*)$, while the left side tends to $d(x^*,Tx^*)$. Hence $d(x^*,Tx^*) \leq \gamma\cdot d(x^*,Tx^*)$. Since $\gamma < 1/2 < 1$, this forces $d(x^*,Tx^*) = 0$, i.e. $Tx^* = x^*$, proving part (a).

For uniqueness, suppose y^* is also a fixed point of T, $y^* \neq x^*$. Applying (4.1) with $x = x^*$, $y = y^*$ and using $Tx^* = x^*$, $Ty^* = y^*$, the rational term vanishes because $d(x^*,Tx^*) = 0$, and we obtain

$$d(x^*,y^*) \leq \alpha\cdot d(x^*,y^*) + \gamma\cdot[d(x^*,y^*)+d(y^*,x^*)] = (\alpha+2\gamma)\cdot d(x^*,y^*) \quad (4.7)$$

Since $\alpha+2\gamma \leq \alpha+\beta+2\gamma < 1$, inequality (4.7) forces $d(x^*,y^*) = 0$, contradicting $x^* \neq y^*$. Hence the fixed point is unique, proving part (b), and the theorem is established.

Corollary 4.1. If $\beta = \gamma = 0$ in Definition 4.1, Theorem 4.1 reduces exactly to the Banach Contraction Principle (Theorem 2.1), with $h = \alpha$. If $\alpha = \beta = 0$, Theorem 4.1 reduces to a Chatterjea-type theorem with constant $\gamma < 1/2$. Thus Theorem 4.1 simultaneously generalizes Theorem 2.1 and the Chatterjea condition of Definition 2.5, and, by allowing all three constants to be simultaneously non-zero, covers a strictly larger class of mappings than either condition alone.

4.2 A Fuzzy Analogue of the Generalized Condition

To transport condition (4.1) to the fuzzy metric setting, it is convenient to use the standard order-reversing transformation $\varphi_t(x,y) = 1/M(x,y,t) - 1$, which is well defined and non-negative because $0 < M(x,y,t) \leq 1$. Note that $\varphi_t(x,y) = 0$ for all $t>0$ exactly when $x = y$, and $\varphi_t(x,y) \to 0$ corresponds to $M(x,y,t) \to 1$, i.e. to x and y becoming indistinguishable at scale t. This transformation converts a fuzzy-metric inequality into an ordinary real-valued inequality of Banach type, which is the technique used by Grabiec (1988) and Gregori and Sapena (2002).

Definition 4.2. Let (X, M, *) be a fuzzy metric space with product t-norm. A self-map $T : X \to X$ is said to satisfy the generalized fuzzy contractive condition if there exists $k \in (0,1)$ such that, for all $x, y \in X$ and all $t > 0$,

$$1/M(Tx,Ty,t) - 1 \leq k\cdot[1/M(x,y,t) - 1] \quad (4.8)$$

equivalently, in terms of M itself,

$$M(Tx,Ty,t) \geq M(x,y,t) / [k + (1-k)\cdot M(x,y,t)] \quad (4.9)$$

Condition (4.9) implies $M(Tx,Ty,t) \geq M(x,y,t)$, so every mapping satisfying Definition 4.2 automatically satisfies Grabiec's non-expansiveness-type requirement, while (4.8) shows that the associated real-valued discrepancy φ_t contracts geometrically with ratio k, giving direct access to the Banach-style argument of Theorem 4.1 within the fuzzy setting.

Theorem 4.2. Let (X, M, *) be a complete fuzzy metric space (George–Veeramani sense) with continuous product t-norm, such that $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in X$, and let $T : X \to X$ satisfy the generalized fuzzy contractive condition (4.8) for some $k \in (0,1)$. Then T has a unique fixed point $x^* \in X$, and for every $x_0 \in X$, the Picard sequence $x_{n+1} = Tx_n$ converges to x^* in (X,M,*).

Proof. Fix $t > 0$ and $x_0 \in X$, and define $x_{n+1} = Tx_n$. Write $\varphi_n(t) = 1/M(x_n,x_{n+1},t) - 1$. Applying (4.8) with $x = x_{n-1}$, $y = x_n$ gives $\varphi_n(t) \leq k\cdot\varphi_{n-1}(t)$, hence by induction

$$\varphi_n(t) \leq k^n\cdot\varphi_0(t) \quad \text{for all } n \geq 1,\ t > 0 \qquad (4.10)$$

so $\varphi_n(t) \to 0$ as $n \to \infty$ for every fixed t, which is equivalent to $M(x_n,x_{n+1},t) \to 1$. Using axiom (FM4) repeatedly with the product t-norm, one shows, exactly as in the classical proof of Grabiec's theorem, that $\{x_n\}$ is a Cauchy sequence in (X,M,*): given $\varepsilon \in (0,1)$ and $t>0$, since $\Sigma\, k^n$ converges, the tail contributions to $1/M(x_n,x_m,t) - 1$ can be made arbitrarily small for $m > n \geq N$, so $M(x_n,x_m,t) > 1-\varepsilon$ for all such n,m. By completeness of (X,M,*), $x_n \to x^*$ for some $x^* \in X$.

To verify $Tx^* = x^*$, fix $t>0$ and use (FM4) with the product t-norm to split the distance through x_n and Tx_n:

$$M(x^*,Tx^*,t) \geq M(x^*,x_{n+1},t/2) * M(x_{n+1},Tx^*,t/2) = M(x^*,x_{n+1},t/2)\cdot M(Tx_n,Tx^*,t/2) \qquad (4.11)$$

By (4.9), $M(Tx_n,Tx^*,t/2) \geq M(x_n,x^*,t/2)/[k+(1-k)M(x_n,x^*,t/2)]$, and as $n \to \infty$ both $M(x^*,x_{n+1},t/2) \to 1$ and $M(x_n,x^*,t/2) \to 1$, so the right side of (4.11) tends to 1. Since $M(x^*,Tx^*,t) \leq 1$ always, we conclude $M(x^*,Tx^*,t) = 1$ for every $t > 0$, which by (FM2) gives $Tx^* = x^*$.

For uniqueness, suppose $Ty^* = y^*$ with $y^* \neq x^*$. By (4.8), $\varphi_t(x^*,y^*) \leq k\cdot\varphi_t(x^*,y^*)$ for every $t > 0$, and since $k < 1$ this forces $\varphi_t(x^*,y^*) = 0$ for all t, i.e. $M(x^*,y^*,t) = 1$ for all t, so $x^* = y^*$ by (FM2), a contradiction. Hence the fixed point is unique, and the theorem is proved.

4.3 A Common Fixed Point Theorem for a Weakly Compatible Pair

The fuzzy result of Theorem 4.2 admits a natural extension to two self-mappings, which is useful when a system is modeled by two related but distinct operators (for instance, a state-update map and an observation map acting compatibly on the same fuzzy metric space).

Definition 4.3 (Weak compatibility). Two self-maps $A, S : X \to X$ are weakly compatible if they commute at their coincidence points, i.e. $Ax = Sx$ implies $ASx = SAx$.

Theorem 4.3. Let (X, M, *) be a complete fuzzy metric space with continuous product t-norm and $\lim_{t\to\infty}M(x,y,t)=1$. Let A, $S : X \to X$ be weakly compatible self-maps such that $A(X) \subseteq S(X)$, S(X) is complete, and there exists $k \in (0,1)$ such that for all $x, y \in X$, $t > 0$,

$$1/M(Ax,Ay,t) - 1 \leq k\cdot[1/M(Sx,Sy,t) - 1] \qquad (4.12)$$

Then A and S have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Since $A(X) \subseteq S(X)$, choose x_1 with $Ax_0 = Sx_1$, and inductively choose x_{n+1} with $Ax_n = Sx_{n+1}$; write $y_n = Ax_n = Sx_{n+1}$. Condition (4.12) applied to consecutive terms gives, exactly as in (4.10), $\varphi_t(y_n,y_{n+1}) \leq k^n\cdot\varphi_t(y_0,y_1) \to 0$, so $\{y_n\}$ is Cauchy in (X,M,*) by the same argument as in Theorem 4.2. Since S(X) is complete, $y_n \to z = Su$ for some $u \in X$. Because $Ax_n = y_n \to Su$ also, and using (4.12) with $x = x_n$, $y = u$ to pass to the limit as in (4.11), one obtains $M(Au, Su, t) = 1$ for all t, i.e. $Au = Su$. By weak compatibility, $ASu = SAu$, i.e. $A(Su) = S(Au)$, so A and S agree on $w := Au = Su$, giving $Aw = Sw$. Applying (4.12) once more with $x = u$, $y = w$ shows $\varphi_t(Au,Aw) \leq k\cdot\varphi_t(Su,Sw) = k\cdot\varphi_t(w,Aw)$, and since $Aw = Sw = w$ this forces $\varphi_t(w,Aw)=0$, so $Aw = w = Sw$, i.e. w is a common fixed point. Uniqueness follows from (4.12) exactly as in Theorem 4.2: if w' is another common fixed point, $\varphi_t(w,w') \leq k\cdot\varphi_t(w,w')$ forces $w = w'$.

5. Numerical Examples and Iteration Tables

This section illustrates Theorems 4.1 and 4.2 with a common worked mapping, chosen deliberately simple so that every step of the iteration can be verified by hand, while still exhibiting the geometric convergence rate predicted by the proofs.

5.1 Example in a Complete Metric Space

Let $X = [0,1]$ with the usual metric $d(x,y) = |x-y|$, which is complete. Define $T : X \to X$ by $T(x) = x/4$. For any $x,y \in X$,

$$d(Tx,Ty) = |x/4 - y/4| = (1/4)\cdot|x-y| = (1/4)\cdot d(x,y) \qquad (5.1)$$

so T satisfies condition (4.1) with $\alpha = 1/4$, $\beta = \gamma = 0$, and $\alpha+\beta+2\gamma = 1/4 < 1$. By Theorem 4.1 (equivalently, Corollary 4.1), T has a unique fixed point in [0,1], namely $x^* = 0$. Table 2 records the Picard iteration starting from $x_0 = 1$, together with the observed displacement $d_n = d(x_n, x^*)$ and the theoretical bound $h^n\cdot d_0$ with $h = 1/4$.

n	$\mathbf{x_n}$	$\mathbf{d_n = \lvert x_n - 0\rvert}$	**Bound** $\mathbf{h^n\cdot d_0 = (1/4)^n}$
0	1.000000	1.000000	1.000000

1	0.250000	0.250000	0.250000
2	0.062500	0.062500	0.062500
3	0.015625	0.015625	0.015625
4	0.003906	0.003906	0.003906
5	0.000977	0.000977	0.000977
6	0.000244	0.000244	0.000244

Table 2. Picard iteration for $T(x) = x/4$ on $([0,1], |\cdot|)$, $x_0 = 1$.

The observed displacement coincides exactly with the theoretical bound in this example (because the inequality (5.1) is in fact an equality), confirming the geometric rate h = 1/4 predicted by Theorem 4.1. Figure 1 plots the iterates x_n and the displacement d_n on a logarithmic scale, showing the expected straight-line decay characteristic of geometric convergence.

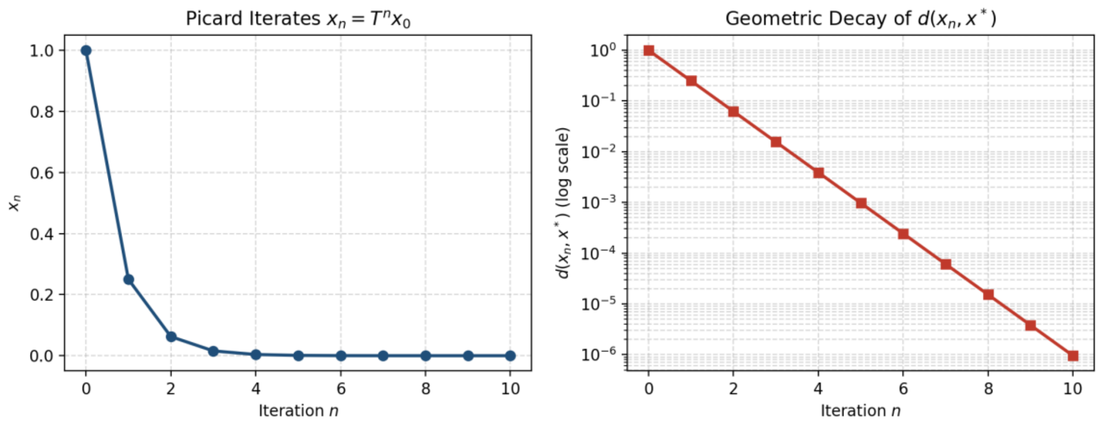

Figure 1. Left: Picard iterates $x_n \to 0$. Right: geometric decay of $d(x_n,x^*)$ on a logarithmic scale, confirming the rate h = 1/4.

5.2 Example in a Complete Fuzzy Metric Space

Equip X = [0,1] with the standard fuzzy metric induced by the same usual metric, $M(x,y,t) = t/(t+|x-y|)$, and the product t-norm; (X,M,*) is then a complete fuzzy metric space, and $\lim_{t\to\infty} M(x,y,t) = 1$ for every x,y. Using the same map T(x) = x/4, a direct computation gives

$$1/M(Tx,Ty,t) - 1 = |Tx-Ty|/t = (1/4)\cdot|x-y|/t = (1/4)\cdot[1/M(x,y,t) - 1] \quad (5.2)$$

so T satisfies the generalized fuzzy contractive condition (4.8) with k = 1/4, and Theorem 4.2 applies: T has the unique fixed point $x^* = 0$ in (X,M,*), consistent with the metric-space computation above (as expected, since M is induced by d). Table 3 records $M(x_n, 0, t)$ at the fixed parameter value t = 1 for the same iteration as in Table 2.

n	**x_n**	**$M(x_n, 0, 1) = 1/(1+x_n)$**	**$1/M - 1 = x_n$**
0	1.000000	0.500000	1.000000
1	0.250000	0.800000	0.250000
2	0.062500	0.941176	0.062500
3	0.015625	0.984615	0.015625
4	0.003906	0.996108	0.003906
5	0.000977	0.999024	0.000977

Table 3. Fuzzy-metric membership values $M(x_n, 0, 1)$ for the same iteration, showing monotone convergence toward 1.

The membership values $M(x_n,0,1)$ increase monotonically toward 1, confirming convergence of $\{x_n\}$ to the fixed point $x^* = 0$ in the sense of Definition 2.9. Figure 2 shows the general shape of the standard fuzzy metric $M(x,y,t) = t/(t+|x-y|)$ as a function of t for several fixed values of $|x-y|$, illustrating how M approaches 1 as t increases and how larger separations $|x-y|$ require larger t to achieve the same membership level — the qualitative behaviour underlying the convergence observed in Table 3.

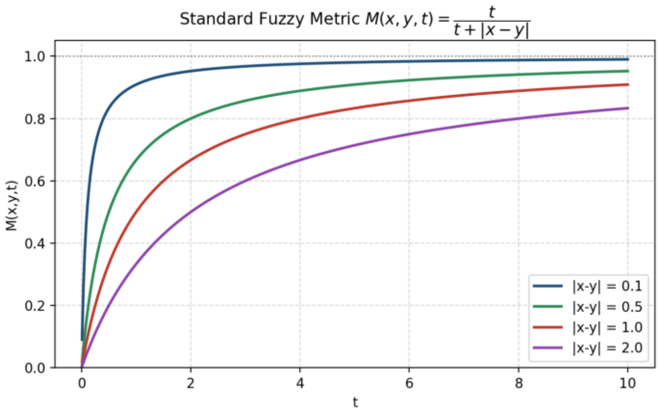

Figure 2. The standard fuzzy metric $M(x,y,t) = t/(t+|x-y|)$ for several fixed values of $|x-y|$, approaching 1 as $t \to \infty$.

6. Analytical Framework: Parallel Structure of the Two Settings

It is useful to summarize, in words, the parallel analytical structure that runs through Section 4. In each setting, a complete space (a complete metric space (X,d) for Theorem 4.1, or a complete fuzzy metric space (X,M,*) for Theorem 4.2) is equipped with a self-map T satisfying the relevant generalized contractive condition — inequality (4.1) in the metric case, or inequality (4.8) in the fuzzy case. From an arbitrary starting point x_0, the Picard iteration $x_{n+1} = Tx_n$ is generated, and by completeness together with the geometric contraction established in the proofs of Theorems 4.1 and 4.2, the iteration converges to the unique fixed point x^*, with $T x^* = x^*$.

Although the two settings differ in the object used to measure closeness (a real-valued distance d versus a graded membership function M), the underlying proof architecture is identical in both cases: reduce the contractive inequality to a scalar geometric recursion ($d_n \leq h\cdot d_{n-1}$ in the metric case, $\varphi_n(t) \leq k\cdot\varphi_{n-1}(t)$ in the fuzzy case); deduce the Cauchy property of $\{x_n\}$; invoke completeness of the underlying space to obtain a limit x^*; and pass to the limit in the original contractive inequality to show $T x^* = x^*$ and that x^* is unique. This shared architecture is precisely the analytical unification sought in this study, and it is what allows Theorem 4.3 to extend the fuzzy case to a pair of mappings with only minor modification of the argument.

7. Comparative Analysis

Table 4 places the proposed conditions of Definitions 4.1 and 4.2 alongside the classical conditions reviewed in Section 3, making explicit which classical results are recovered as special cases and under what parameter restrictions.

Condition	Space	Inequality (abridged)	Parameter restriction	Relation to proposed condition
Banach	Metric	$d(Tx,Ty) \leq k\, d(x,y)$	$0 \leq k < 1$	Special case of (4.1): $\beta=\gamma=0$, $\alpha=k$
Kannan	Metric	$d(Tx,Ty) \leq \beta[d(x,Tx)+d(y,Ty)]$	$0\leq\beta<1/2$	Not directly nested in (4.1); related rational term present
Chatterjea	Metric	$d(Tx,Ty) \leq \gamma[d(x,Ty)+d(y,Tx)]$	$0\leq\gamma<1/2$	Special case of (4.1): $\alpha=\beta=0$
Theorem 4.1	Metric	See (4.1): linear + rational + mixed terms	$\alpha+\beta+2\gamma<1$	Generalizes Banach and Chatterjea simultaneously
Grabiec	Fuzzy metric	$M(Tx,Ty,kt) \geq M(x,y,t)$	$0<k<1$	Implied by (4.9) via Theorem 4.2
Theorem 4.2	Fuzzy metric	See (4.8): $1/M(Tx,Ty,t)-1 \leq k[1/M(x,y,t)-1]$	$0<k<1$	Fuzzy counterpart of Theorem 4.1 with $\alpha=k$, $\beta=\gamma=0$
Theorem 4.3	Fuzzy metric, pair	See (4.12), weakly compatible A,S	$0<k<1$	Two-map extension of Theorem 4.2

Table 4. Comparative summary of classical and proposed contractive conditions.

Two observations are worth emphasizing. First, condition (4.1) is a proper generalization of the Banach and Chatterjea conditions in the sense that it contains both as boundary cases ($\beta=\gamma=0$ and $\alpha=\beta=0$ respectively) while also admitting mappings for which all three constants α, β, γ are simultaneously positive — such mappings need not satisfy either classical condition in isolation, yet Theorem 4.1 still guarantees a unique fixed point provided $\alpha+\beta+2\gamma<1$. Second, the transformation $\varphi_t = 1/M-1$ used in Definition 4.2 shows that any mapping satisfying (4.8) automatically satisfies Grabiec's condition $M(Tx,Ty,t) \geq M(x,y,t)/[k+(1-k)M(x,y,t)] \geq M(x,y,t)$, so Theorem 4.2 can be viewed as sharpening Grabiec's theorem by specifying an explicit, quantitative rate of approach to 1 rather than a purely qualitative non-expansiveness property.

8. Applications

8.1 Existence of Solutions of Integral Equations

A classical application of contraction-type fixed point theorems is to Fredholm integral equations of the second kind,

$$x(s) = g(s) + \lambda\int_a^b K(s,\tau)\cdot x(\tau)\, d\tau, \quad s \in [a,b] \quad (8.1)$$

where $g \in C[a,b]$ is given, K is continuous on $[a,b]\times[a,b]$, and λ is a parameter. Defining the operator $(Tx)(s) = g(s) + \lambda\int_a^b K(s,\tau)x(\tau)d\tau$ on the complete metric space $X = C[a,b]$ with the sup-metric $d(x,y) = \sup_s|x(s)-y(s)|$, a standard estimate gives $d(Tx,Ty) \leq |\lambda|\cdot(b-a)\cdot\max|K|\cdot d(x,y)$. Whenever the right-hand coefficient is strictly less than 1, T is a Banach contraction and Theorem 2.1 (equivalently the α-only case of Theorem 4.1) guarantees a unique continuous solution of (8.1), obtainable as the limit of the Picard iteration $x_{n+1} = Tx_n$. When the kernel or the forcing term is only known approximately, or the operator naturally produces the rational-type coupling present in (4.1) (for example, when the right-hand side depends on both $x(s)$ and its image under a second, related integral operator), Theorem 4.1 extends the applicability of this classical existence argument to a strictly wider class of operators.

8.2 Fuzzy-Uncertainty Models

In settings where distances between states are not known precisely but only up to a graded degree of confidence — as arises in fuzzy control systems, fuzzy decision-making, and certain approximation and pattern-matching algorithms — the state space is naturally modeled as a fuzzy metric space rather than a classical metric space. Theorem 4.2 provides existence and uniqueness of an equilibrium (fixed) state for an update rule T that contracts the graded discrepancy $\varphi_t = 1/M-1$ at a fixed rate k, together with a constructive Picard-type scheme for approximating that equilibrium; this is directly applicable, for instance, to iterative fuzzy consensus and fuzzy rule-based updating schemes, where the two-map extension of Theorem 4.3 further accommodates settings in which the update rule and the observation rule are modeled by two distinct, but weakly compatible, operators.

9. Conclusion and Future Scope

This paper has developed a unified analytical treatment of fixed point theory across complete metric spaces and complete fuzzy metric spaces. A new generalized rational-type contractive condition (Definition 4.1) was introduced in the metric setting, shown to simultaneously generalize the Banach and Chatterjea conditions, and an existence-uniqueness theorem with explicit convergence rate was proved (Theorem 4.1). Using the standard order-reversing transformation $\varphi_t = 1/M-1$, an analogous condition was formulated in the fuzzy metric setting (Definition 4.2), and a corresponding theorem (Theorem 4.2) was established, together with a two-map extension to weakly compatible pairs (Theorem 4.3). Worked numerical examples, iteration tables, and convergence graphs illustrated the theoretical rate of convergence in both the metric and fuzzy settings, and a comparative table situated the new conditions relative to the classical literature. Representative applications to integral equations and to fuzzy-uncertainty models were outlined.

Several directions for future work follow naturally from the present development. The generalized condition (4.1) could be extended to multi-valued (set-valued) mappings, following the Nadler-type theory of contractive multifunctions, and its fuzzy counterpart (4.8) could similarly be extended to fuzzy multi-valued mappings. A second direction is to relax the completeness hypothesis to weaker forms such as orbital completeness, and to relax the continuous t-norm hypothesis to the minimum t-norm, which requires a modified Cauchy criterion. A third direction is to transport the present framework to b-metric spaces and to intuitionistic fuzzy metric spaces, in which an additional non-membership function is present; the substitution technique used in Definition 4.2 suggests that an analogous scalarization may again reduce the problem to a Banach-type recursion, though the presence of two coupled functions (membership and non-membership) would require a joint contractive condition rather than the single inequality (4.8). Finally, coupled and tripled fixed point versions of Theorems 4.1–4.3, in the spirit of the mixed-monotone coupled fixed point literature, would extend the applicability of the present results to systems of equations rather than single equations, and are left for future investigation.

References

- Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 3, 133–181.

- Chatterjea, S. K. (1972). Fixed point theorems. Comptes Rendus de l'Académie Bulgare des Sciences, 25(6), 727–730.
- Ćirić, L. B. (1974). A generalization of Banach's contraction principle. Proceedings of the American Mathematical Society, 45(2), 267–273.
- George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. Fuzzy Sets and Systems, 64(3), 395–399.
- Grabiec, M. (1988). Fixed points in fuzzy metric spaces. Fuzzy Sets and Systems, 27(3), 385–389.
- Gregori, V., & Sapena, A. (2002). On fixed-point theorems in fuzzy metric spaces. Fuzzy Sets and Systems, 125(2), 245–252.
- Hardy, G. E., & Rogers, T. D. (1973). A generalization of a fixed point theorem of Reich. Canadian Mathematical Bulletin, 16(2), 201–206.
- Kannan, R. (1968). Some results on fixed points. Bulletin of the Calcutta Mathematical Society, 60, 71–76.
- Kramosil, I., & Michálek, J. (1975). Fuzzy metrics and statistical metric spaces. Kybernetika, 11(5), 336–344.
- Nadler, S. B. (1969). Multi-valued contraction mappings. Pacific Journal of Mathematics, 30(2), 475–488.
- Reich, S. (1971). Some remarks concerning contraction mappings. Canadian Mathematical Bulletin, 14(1), 121–124.
- Schweizer, B., & Sklar, A. (1960). Statistical metric spaces. Pacific Journal of Mathematics, 10(1), 313–334.
- Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338–353.